The Foucault Pendulum - a Simplified Trajectory Analysis for a Pendulum on a Turntable and an Outlook to a Pendulum on Earth

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1 Avant – Propos

The following text is an update of a treatise written in 2008 and titled "The Foucault Pendulum's Trajectory – the Formalism". That text contained a subtle and for long unnoticed – let's say – imperfection which was detected and communicated by a sharp-eyed observer from outside, see below. Since this error – possibly due to a custom of tradition in the word's genuine meaning – is found in several articles on the slowly swinging device, the authors of the present text have made an attempt to illuminate the Foucault Pendulum's (FKP's) properties from as many views as possible. This implies some mathematics concerned with rotating coordinate frames and might raise unpleasant emotions in one or the other reader.

Regarding this formalism as indispensible, it will be presented here in full beauty. But, to describe the physics essentials with a minimum of torture, the first paragraphs concerned with the formalism will presented and explained in detail such that any reader still familiar with high school mathematics can enjoy it – hopefully. But let us start with a short introduction.

2 Introducing the Pendulum

2.1 Assumptions and Approximations

A Foucault pendulum is, at first sight, a rather simple device: Quite a normal pendulum, driven by gravity and inertia, whose plane of oscillation moves clockwise – at least in the northern hemisphere of our planet – due to the Earth's proper rotation about its own axis.

But at a closer look, there are some sources of perturbation and distortion which we will shortly mention here but rather neglect when discussing the FKP's oscillation period and trajectory.

First, when the pendulum swings towards the position of maximum amplitude, it will be lifted a bit so that it gains some potential energy – else there would be no retracting force at all. The pendulum moves in three-dimensional space. But we will treat it as moving only in a plane, which is justified by the smallness of the vertical displacement.

Second, the pendulum rope, 11 m long in the KIP case, can easily be excited to small transverse oscillations. These, together with tiny perturbations due to mechanical imperfections, can cause the pendulum to deviate fom planar swinging and start a rather elliptical path such that the ellipse performs sort of "Lissajous motion" which is not caused by the Coriolis force, but is rather due to phase differences between the movements of small and big half-axis. These deviations fom planar swinging can, however, effectively reduced by a small ring ("Charron ring") around the pendulum rope mounted close to the rope's fixation point.
Finally, our device is treated as a "mathematical pendulum", i.e. a point mass hanging on a massless, infinitely thin rope. This is again (rather) justified by the dimensions of pendulum bob and rope. In addition we assume a non-damped motion of the pendulum bob.

In short: The results derived below are somewhat idealised as the above mentioned effects are not taken into account which we regard as justified due to their smallnes.

2.2 Towards the Formalism

The pendulum which can be admired e.g. in the Kirchhoff-Institut in Heidelberg moves in the real world, i.e. it experiences forces from the Earth's gravitational field: our planet possesses a mass of 5.973×10^{24} kg, leading to the acceleration of $g = 9.81\,\mathrm{m/s^2}$. The gravitational force and the pendulum's inertia keep it swinging.

Since the Earth is – as we believe and will see – a rotating coordinate frame, the pendulum is subject to inertial forces immanent in rotating systems. The first and very familiar one is the centrifugal force: A physical body moving on a curved trajectory refuses to change its direction – it wants, following Newton's law of inertia, to proceed along the tangent of its path. This results in a force driving it towards the outside of its trajectory – an experience made by any car driver steering her or his vehicle with a speed too high through a narrow bend. On the rotating Earth, also a body at rest feels the centrifugal force. The force counterbalancing the centrifugal one is called centripetal force – in case of the fast car it is provided by tyres connecting it to the road.

Another – and less familiar – inertial force is experienced only by physical bodies *moving* in rotating coordinate frames: The Coriolis force.

2.3 A Closer Look at the Forces Acting

Let's have a short look at the three dominating forces in detail: Gravitational force, directed to a point inside our planet which is determined by the distribution of masses "felt" by the pendulum, can be decomposed into a force acting along the rope – a component which we are not interested in – and a force tangential to the circular path of the pendulum bob. This component always works retracting, i.e. towards the rest position of the pendulum:

2.3.1 The Gravitational Force

Suppose we have a pendulum hanging from a rope of length l, in its equilibrium position. Then, moving it out of this position by an angle ϕ, gravity will try to move it back with a component of $-mg\sin(\phi)$. Here m is the mass of the pendulum bob, and g is the gravitational acceleration at the point where the pendulum is situated. If the displacement is small when compared to the length of the rope, we are allowed to use the "small angle approximation", i.e. $\sin(\phi) \approx \phi$, and the retracting force will be $-mg\phi$. From this we get immediately the equation of motion due to gravitation:

$$ml\ddot{\phi} + mg\phi = 0 \tag{1}$$

We note here that, in the small angle approximation, we may set $l\ddot{\phi} = \ddot{x}$ if, for example, we let the pendulum swing along an x-axis. We will adopt this notation later. Remember: $\ddot{\phi} = \partial^2\phi/\partial t^2$.

We know from our high school days that the solution of this differential equation is a harmonic oscillation with frequency/angular velocity

$$\omega_0 = 2\pi/T_0 = \sqrt{g/l}.$$

Here, T_0 is the pendulum's swinging period, g the gravitational acceleration at the spot where the pendulum is situated, an l the length of the pendulum rope.

2.3.2 The Centrifugal Force

The centrifugal force depends on the angular velocity with which the pendulum plane is rotating. This velocity is very small: Situated at one of the poles, it takes the pendulum 24 hours for a complete circle of 360° or 2π. The modulus of the centrifugal force reads

$$F_{\mathrm{Cen}} = m\Omega^2 R$$

where m is the mass of the pendulum bob, Ω the above mentioned angular velocity of the Earth, and R the distance of the bob from the rotation axis of the rotating frame i.e. the Earth axis. The centrifugal force drives the pendulum bob to the outside, in the direction of growing R.

2.3.3 The Coriolis Force

The Coriolis force, in vector notation, reads

$$\vec{F}_{\text{Cor}} = 2m\vec{v} \times \vec{\Omega}.$$

Here, $\vec{v}$ is the velocity of the pendulum bob (after our simplification in two dimensions only!), and $\vec{\Omega}$ is the vector of the Earth's angular velocity whose modulus is $2\pi/24h$, corresponding to $360°/24h$ when expressed in degrees.

However, the effective angular velocity of our planet which enters the formula for the Coriolis force gets smaller when we move from one of the poles towards the equator: The "vertical" component at a place with latitude θ is only $\omega_E \sin\theta$ so that we have to write our formula for an arbitrary latitude as

$$\vec{F}_{\text{Cor}} = 2m\vec{v} \times \vec{\Omega}_{E,\theta}$$

with the modulus

$$F_{\text{Cor}} = 2mv\Omega_E \sin(\theta).$$

In the pendulum case, the angle between $\vec{v}$ and $\vec{\omega}_E$ is always $90°$ as long as $\vec{v}$ is constraint to the Earth's surface. For a terrestrial observer, the Coriolis force makes the plane in which the pendulum swings rotate clockwise e.g. at the North Pole, and counter-clockwise at the South Pole while it vanishes at the equator. (The Coriolis force is a rather well known phenomenon to the TV watchers looking at the weather report: Storms on the Earth's northern hemisphere always turn to the right, those on the southern hemisphere to the left.)

Now we put things together and sort according to components x and y. Our frame of reference is our laboritory where we observe the pendulum's enigmatic movements – fixed upon the Earth's surface –, and we are free to choose the x-axis e.g. parallel to a meridian, pointing to the North Pole, and the y-axis pointing east along a circle of latitude. Then, with ω_0 as the pendulum's circular frequency, we may write the components of our forces as follows:

$$F_{\text{Grav,x}} = -m\omega_0^2 x,$$
$$F_{\text{Grav,y}} = -m\omega_0^2 y,$$

$$F_{\text{Cen,x}} = m\Omega^2 x,$$
$$F_{\text{Cen,y}} = m\Omega^2 y,$$

$$F_{\text{Cor,x}} = 2m\Omega\dot{y}\sin(\theta),$$
$$F_{\text{Cor,y}} = -2m\Omega\dot{x}\sin(\theta).$$

The first two equations apply to gravitation, the second two apply to the centrifugal force, and the last two to the Coriolis force. The factor $\sin\theta$ accounts for the angle of latitude which is zero at the equator and $90°$ at the north pole. But to simplify calculations (and intuition) we will drop the θ dependence which means that our pendulum moves in a

two-dimensional space touching our planet at (e.g.) the North Pole. It is easy to extend these considerations to an arbitrary spot on earth.
Now we get immediately two differential equations for the x- and y-components of the pendulum bob's trajectory:

$$\ddot{x} = -\omega_0^2 x + \Omega^2 x + 2\Omega\dot{y},$$
$$\ddot{y} = -\omega_0^2 y + \Omega^2 y - 2\Omega\dot{x},$$

So far the introductory remarks meant to describe the physical basis on which one proceeds to derive the equations for the pendulum's trajectory. The mathematical formalism needed for this task will be presented in detail in the following paragraphs analysing the situation for a pendulum on a turntable.

3 The Choice of Frame of Reference and Coordinate Systems for Modelling the Pendulum on a Turntable

For the modelling of the Foucault pendulum there are two apparent possibilities for the frame of reference: We can derive and solve the equation of motion in the rotating and therefore accelerated system of reference K – the "terrestrial" system – or in an inertial frame of reference K' in which the turntable is rotating. We call this system "cosmic system" because we look at our pendulum from a cosmically fixed inertial frame. We indicate coordinates in the inertial frame of reference with a prime. From a mathematical viewpoint the solution of the Foucault problem in the cosmically fixed inertial frame, followed by a simple coordinate transformation, is the easiest way. Unfortunately our every day expierence as residents on the rotating terrestrial system makes us believe that the solution in that frame of reference is easier to achieve. So we start with the treatment of the Foucault problem in this frame of reference. In addition, we have the choice between Cartesian and polar coordinates which for sure does not influence the frame of reference but the way of solving the problem. We will show the situation both Cartesian and polar.

4 The Mathematical Pendulum in an Inertial Frame of Reference

Let us consider the pendulum introduced in Section 2 anew, now using the adopted notation with primed coordinates indicating calculations in an inertial frame and non-primed ones for the rotating frame of reference. With the elongation angle ϕ' in the frame of reference K' we rewrite the differential equation (1) in the following form

$$ml\ddot{\phi}' + mg\phi' = 0 \tag{2}$$

or equivalently

$$l\ddot{\phi}' + g\phi' = 0 \quad . \tag{3}$$

For example, we let the pendulum swing in x'-direction with $x' = l\phi'$ and get the equation of motion

$$\ddot{x}' + \frac{g}{l}x' = 0 \quad . \tag{4}$$

Using the abbreviation $\omega_0^2 = g/l$ we finally get the simple differential equation

$$\ddot{x}' + \omega_0^2 x' = 0. \tag{5}$$

A special solution of the differential equation is given by $x'(t) = x'_0 \cos(\omega_0 t)$ with an angular frequency of $\omega_0 = 2\pi/T_0 = \sqrt{g/l}$ where T_0 denotes the pendulum period. Here at $t = 0$ we start in the elongated state. Swinging in x'-direction is only one possibility of our pendulum's orbit. No direction is preferred. The general equation of motion for a plane two-dimensional mathematical pendulum in the small angle approximation is given by

$$\begin{pmatrix} \ddot{x}' \\ \ddot{y}' \end{pmatrix} + \omega_0^2 \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad . \tag{6}$$

Eigensolutions to this differential equation are two circular orbits of the pendulum bob of the form

$$\vec{R}'_1 = R'_{01} \begin{pmatrix} \cos(\omega_0 t + A) \\ \sin(\omega_0 t + A) \end{pmatrix} \tag{7}$$

and

$$\vec{R}'_2 = R'_{02} \begin{pmatrix} \cos(\omega_0 t + B) \\ -\sin(\omega_0 t + B) \end{pmatrix} \quad . \tag{8}$$

The amplitudes R'_{01}, R'_{02} and phases A, B are given by the starting conditions of the pendulum. For simplification we set the initial phases to zero:

$$\vec{R}'_1 = R'_{01} \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix} \tag{9}$$

$$\vec{R}'_2 = R'_{02} \begin{pmatrix} \cos(\omega_0 t) \\ -\sin(\omega_0 t) \end{pmatrix} \quad . \tag{10}$$

Using these two eigensolutions, we can describe all possible orbits of the pendulum. As an example we calculate the sum

$$\vec{R}'_3 = \vec{R}'_1 + \vec{R}'_2 = R'_{03} \begin{pmatrix} \cos(\omega_0 t) \\ 0 \end{pmatrix} \tag{11}$$

which is the linear eigenmode along the x'-axis as shown in Fig.1. Note that we do not have to consider the case of the conical pendulum since we restrict ourselves to the small angle approximation.

Another at first glance quite odd possibility is an elliptic orbit with a tangent velocity $R'_{04}\Omega$ perpendicular to the semi-major of the ellipse

$$\vec{R}'_4 = R'_{04} \begin{pmatrix} \cos(\omega_0 t) \\ \frac{\Omega}{\omega_0} \sin(\omega_0 t) \end{pmatrix} \tag{12}$$

visualized in Fig. 2. The meaningfulness of this choice will be seen in section 5.2.1.

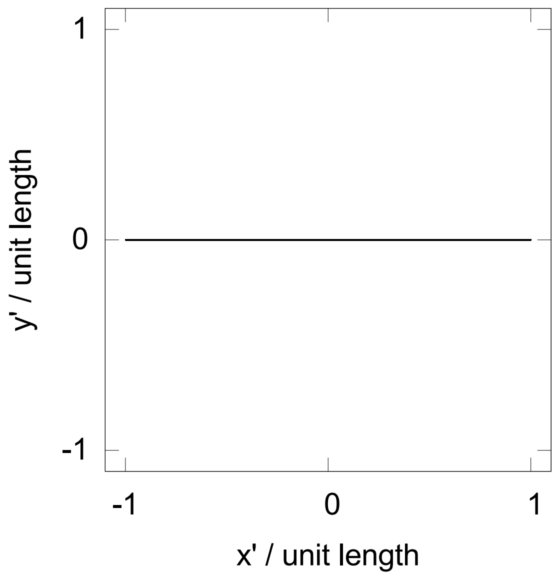

Figure 1 Linear orbit with unit length elongation in x'-direction in the inertial frame of reference.

5 The Foucault Pendulum

5.1 Terrestrial System - Accelerated Frame

In a rotating frame of reference inertial forces are of fundamental importance: The Coriolis force and the centrifugal force. Consequently there are now three forces acting upon the pendulum: Gravity, Coriolis force and centrifugal force. Since force $\vec{F}$ and acceleration $\vec{a}$ according to Newton's law are connected by mass $\vec{F} = m\vec{a}$ and we believe that heavy mass and inert mass is the same, we write down the corresponding accelerations. For simplicity, we discuss a rotating disk with rotation axis parallel to gravity – the "North Pole" situation. The pendulum's pivot point is somewhere along the rotation axis[1]. Hereby whether or not the pivot point is rotating with the frame of reference does not affect the solution of the problem. The disc rotates counterclockwise with an angular frequency Ω with respect to an inertial frame of reference. In vector form this is coded by

$$\vec{\Omega} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix} \quad . \tag{13}$$

The pendulum bob in the plane two-dimensional pendulum approximation moves in the xy-plane with an velocity

$$\vec{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} \quad . \tag{14}$$

[1] "Off-center" pivot points we do not consider here.

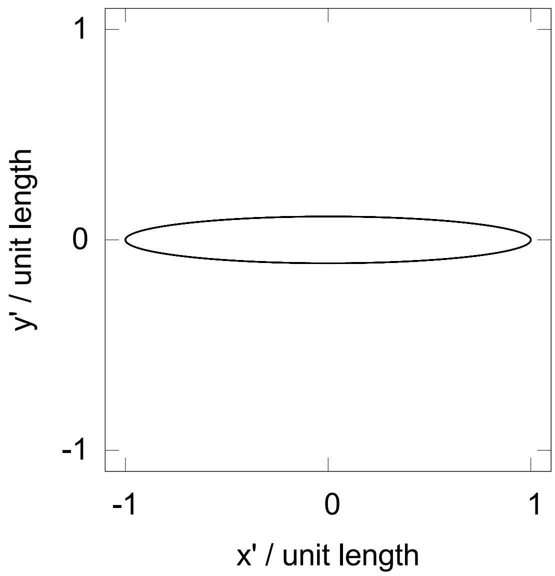

Figure 2 Elliptic orbit with unit length elongation in x'-direction in the inertial frame of reference.

The Coriolis acceleration is given by

$$\vec{a}_{\text{Cor}} = 2\left(\vec{v} \times \vec{\Omega}\right) = 2\begin{pmatrix}\dot{x}\\ \dot{y}\\ 0\end{pmatrix} \times \begin{pmatrix}0\\ 0\\ \Omega\end{pmatrix} = 2\begin{pmatrix}\Omega\dot{y}\\ -\Omega\dot{x}\\ 0\end{pmatrix} \quad . \tag{15}$$

We now calculate the centrifugal contribution

$$\vec{a}_{\text{Cen}} = -\vec{\Omega} \times \left(\vec{\Omega} \times \vec{r}\right) = -\begin{pmatrix}0\\ 0\\ \Omega\end{pmatrix} \times \left(\begin{pmatrix}0\\ 0\\ \Omega\end{pmatrix} \times \begin{pmatrix}x\\ y\\ 0\end{pmatrix}\right) = \begin{pmatrix}\Omega^2 x\\ \Omega^2 y\\ 0\end{pmatrix} \tag{16}$$

where $\vec{r}$ is the position of the pendulum bob. The gravitational acceleration is the same as in the inertial frame and therefore

$$\vec{a}_{\text{Grav}} = -\omega_0^2 \begin{pmatrix}x\\ y\\ 0\end{pmatrix} \quad . \tag{17}$$

Obviously, the z-component vanishes in all accelerations above which allows to write down the differential equation in x and y only:

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} = \begin{pmatrix}\Omega^2 - \omega_0^2 & 0\\ 0 & \Omega^2 - \omega_0^2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}0 & 2\Omega\\ -2\Omega & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} \quad . \tag{18}$$

Following the strategy of solution for the pendulum in the inertial frame and setting possible constant phases to zero, the solution of this differential equation is again a circular orbit, more precisely one clockwise and one counterclockwise orbit with different time constants:

$$\vec{R}_+ = R_+ \begin{pmatrix} \cos((\omega_0 - \Omega)t) \\ \sin((\omega_0 - \Omega)t) \end{pmatrix} \tag{19}$$

and

$$\vec{R}_- = R_- \begin{pmatrix} \cos((\omega_0 + \Omega)t) \\ -\sin((\omega_0 + \Omega)t) \end{pmatrix} \quad . \tag{20}$$

Again suited linear combinations of these two orbits describe all possible orbits of the pendulum bob. As an example, we set $R_+ = R_- = 1/2$ in unit lengths. This leads to the orbit

$$\begin{pmatrix} \frac{1}{2}\cos((\omega_0 - \Omega)t) + \frac{1}{2}\cos((\omega_0 + \Omega)t) \\ \frac{1}{2}\sin((\omega_0 - \Omega)t) - \frac{1}{2}\sin((\omega_0 + \Omega)t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_0 t)\cos(\Omega t) \\ -\cos(\omega_0 t)\sin(\Omega t) \end{pmatrix} \quad . \tag{21}$$

Here the pendulum initially is enlongated say from someone “sitting” in the inertial frame. Let us call this starting condition “cosmic start”. Obviously the pendulum bob initially has a non-vanishing tangential velocity in the rotating frame. The corresponding Foucault rosette is shown in Fig. 3. Here we use $\omega_0/\Omega = 9$ and start at $x = 1, y = 0$. Starting

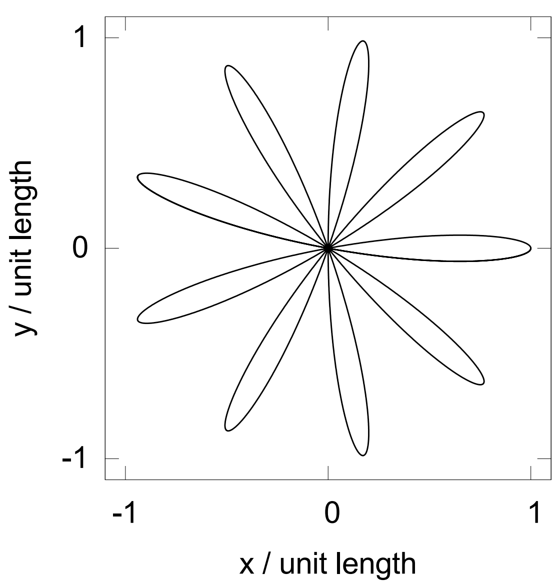

Figure 3 Start of the pendulum elongated cosmically fixed. The corresponding orbit in the inertial frame of reference is shown in Fig. 1.

with vanishing velocities in the rotating frame of reference (“terrestrial start”) requires the following linear combination:

$$\frac{\omega_0 + \Omega}{2\omega_0} \begin{pmatrix} \cos((\omega_0 - \Omega)t) \\ \sin((\omega_0 - \Omega)t) \end{pmatrix} + \frac{\omega_0 - \Omega}{2\omega_0} \begin{pmatrix} \cos((\omega_0 + \Omega)t) \\ -\sin((\omega_0 + \Omega)t) \end{pmatrix} \tag{22}$$

which can be simplified to

$$\begin{pmatrix} \cos(\omega_0 t)\cos(\Omega t) + \frac{\Omega}{\omega_0}\sin(\omega_0 t)\sin(\Omega t) \\ -\cos(\omega_0 t)\sin(\Omega t) + \frac{\Omega}{\omega_0}\sin(\omega_0 t)\cos(\Omega t) \end{pmatrix} \tag{23}$$

visualized in Fig. 4. Again we use $\omega_0/\Omega = 9$ and start at $x = 1, y = 0$ at rest in the

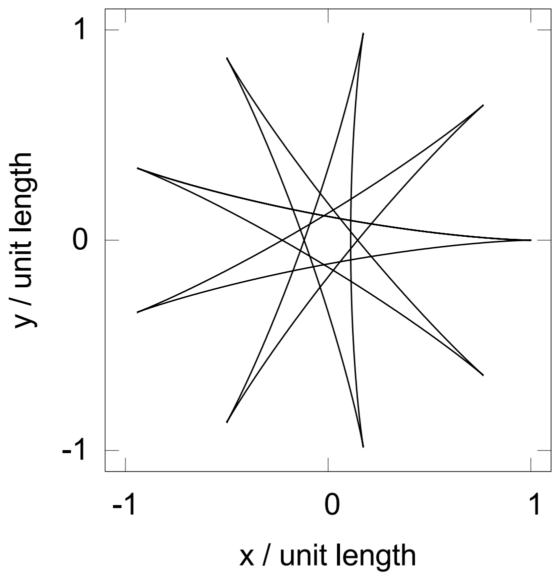

Figure 4 Terrestrial start orbit of the pendulum in the rotating frame of reference. The corresponding orbit in the inertial frame of reference is shown in Fig. 2.

rotating frame of reference.

5.2 Cosmic System – Inertial Frame

5.2.1 Transformation of the Inertial Frame Orbit

Solving the Foucault problem in an inertial frame or, in other words, as viewed from a fixed star is much easier and less complicated since no inertial accelerations have to be taken into account. To transform coordinates into a coordinate system rotated by an angle α, the transformation matrix

$$T_\alpha^{-1} = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix} \tag{24}$$

is used. Again for simplification we assume vanishing phases and set the amplitudes to unity. Now α becomes time dependent: $\alpha = \Omega t$. We apply this transformation to the eigensolutions in the inertial frame

$$\begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) \\ \sin(\omega_0 t) \end{pmatrix} = \begin{pmatrix} \cos((\omega_0 - \Omega)t) \\ \sin((\omega_0 - \Omega)t) \end{pmatrix} \tag{25}$$

$$\begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) \\ -\sin(\omega_0 t) \end{pmatrix} = \begin{pmatrix} \cos((\omega_0 + \Omega)t) \\ -\sin((\omega_0 + \Omega)t) \end{pmatrix} \tag{26}$$

and we are done. The individual trajectory of the pendulum bob again depends on the starting conditions. The oscillation period defined by zero crossings or the time elapsed between two reversal points of the pendulum bob's trajectory is exactly $T_0 = 2\pi/\omega_0$ independent of the frame of reference. Each orbit can be experessed by a linear combination of those two rotated eigensolutions. For sure we can transform also a given orbit in K' individually. Let us take the opportunity and transform

$$\vec{R}'_3 = R'_{03} \begin{pmatrix} \cos(\omega_0 t) \\ 0 \end{pmatrix} \tag{27}$$

and

$$\vec{R}'_4 = R'_{04} \begin{pmatrix} \cos(\omega_0 t) \\ \frac{\Omega}{\omega_0} \sin(\omega_0 t) \end{pmatrix} \tag{28}$$

via

$$\vec{R}_{3,4} = T_\alpha^{-1} \vec{R}'_{3,4}. \tag{29}$$

Explicitely we calculate

$$\vec{R}_3 = R'_{03} \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) \\ 0 \end{pmatrix} = R'_{03} \begin{pmatrix} \cos(\omega_0 t)\cos(\Omega t) \\ -\cos(\omega_0 t)\sin(\Omega t) \end{pmatrix} \quad . \tag{30}$$

Without changing the essential statement we set R'_{03} to unit length and get (21). The same procedure for $\vec{R}'_4$ yields

$$\vec{R}_4 = R'_{04} \begin{pmatrix} \cos(\Omega t) & \sin(\Omega t) \\ -\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \begin{pmatrix} \cos(\omega_0 t) \\ \frac{\Omega}{\omega_0} \end{pmatrix} = R'_{04} \begin{pmatrix} \cos(\omega_0 t)\cos(\Omega t) + \frac{\Omega}{\omega_0}\sin(\omega_0 t)\sin(\Omega t) \\ -\cos(\omega_0 t)\sin(\Omega t) + \frac{\Omega}{\omega_0}\sin(\omega_0 t)\cos(\Omega t) \end{pmatrix} \quad . \tag{31}$$

By setting R'_{04} to unit length we obtain (23).

5.2.2 Transformation of the inertial frame differential equation

Equivalently we can transform the differential equation given in the inertial frame into the rotating one. Following from the transformation matrix T_α^{-1} we get

$$x = x'\cos(\Omega t) + y'\sin(\Omega t) \tag{32}$$

$$y = -x'\sin(\Omega t) + y'\cos(\Omega t) \quad . \tag{33}$$

The derivative with respect to time is given by

$$\dot{x} = \dot{x}'\cos(\Omega t) - x'\Omega\sin(\Omega t) + \dot{y}'\sin(\Omega t) + y'\Omega\cos(\Omega t) \tag{34}$$

$$\dot{y} = -\dot{x}'\sin(\Omega t) - x'\Omega\cos(\Omega t) + \dot{y}'\cos(\Omega t) - y'\Omega\sin(\Omega t) \tag{35}$$

which can be simplified to

$$\dot{x} = \Omega y + \dot{x}'\cos(\Omega t) + \dot{y}'\sin(\Omega t) \tag{36}$$

$$\dot{y} = -\Omega x - \dot{x}'\sin(\Omega t) + \dot{y}'\cos(\Omega t) \tag{37}$$

and the second derivative with respect to time is given by

$$\ddot{x} = \Omega\dot{y} + \ddot{x}'\cos(\Omega t) - \dot{x}'\sin(\Omega t) + \ddot{y}'\sin(\Omega t) + \dot{y}'\Omega\cos(\Omega t) \tag{38}$$

$$\ddot{y} = -\Omega\dot{x} - \ddot{x}'\sin(\Omega t) - \dot{x}'\cos(\Omega t) + \ddot{y}'\cos(\Omega t) - \dot{y}'\Omega\sin(\Omega t) \tag{39}$$

and after simplification and elimination of all primed coordinates

$$\begin{aligned} \ddot{x} &= 2\Omega\dot{y} - \omega_0^2 x + \Omega^2 x & (40) \\ \ddot{y} &= -2\Omega\dot{x} - \omega_0^2 y + \Omega^2 y. & (41) \end{aligned}$$

Finally we get

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \Omega^2 - \omega_0^2 & 0 \\ 0 & \Omega^2 - \omega_0^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 2\Omega \\ -2\Omega & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} \qquad (42)$$

and are done.

5.2.3 Rewriting the Differential Equation in Polar Coordinates

The Cartesian coordinates x and y can be expressed by a pair of polar coordinates r and β, defined by

$$\begin{aligned} x &= r\cos(\beta) & (43) \\ y &= r\sin(\beta) & (44) \end{aligned}$$

where r denotes the radius and β the azimuthal angle. For the first and second derivative with respect to time we calculate

$$\begin{aligned} \dot{x} &= \dot{r}\cos(\beta) - r\dot{\beta}\sin(\beta) & (45) \\ \dot{y} &= \dot{r}\sin(\beta) + r\dot{\beta}\cos(\beta) & (46) \end{aligned}$$

and

$$\begin{aligned} \ddot{x} &= \cos(\beta)\left(\ddot{r} - r\dot{\beta}^2\right) - \sin(\beta)\left(2\dot{r}\dot{\beta} - r\ddot{\beta}\right) & (47) \\ \ddot{y} &= \sin(\beta)\left(\ddot{r} - r\dot{\beta}^2\right) + \cos(\beta)\left(2\dot{r}\dot{\beta} + r\ddot{\beta}\right). & (48) \end{aligned}$$

After inserting these equations into (18) we obtain

$$\begin{aligned} \cos(\beta)\left(\ddot{r} - r\dot{\beta}^2\right) - \sin(\beta)\left(2\dot{r}\dot{\beta} - r\ddot{\beta}\right) &= \left(\Omega^2 - \omega_0^2\right) r\cos(\beta) + 2\Omega\dot{r}\sin(\beta) + 2\Omega r\dot{\beta} \\ \sin(\beta)\left(\ddot{r} - r\dot{\beta}^2\right) + \cos(\beta)\left(2\dot{r}\dot{\beta} + r\ddot{\beta}\right) &= \left(\Omega^2 - \omega_0^2\right) r\sin(\beta) - 2\Omega\dot{r}\cos(\beta) + 2\Omega r\dot{\beta} \quad . \end{aligned}$$

These equations must be valid for all β which implies the following relations resulting from the x-part

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= r\left(\Omega^2 - \omega_0^2\right) + 2\Omega r\dot{\beta} & (49) \\ r\ddot{\beta} - 2\dot{r}\dot{\beta} &= 2\Omega\dot{r} & (50) \end{aligned}$$

and from the y-part

$$\begin{aligned} \ddot{r} - r\dot{\beta}^2 &= r\left(\Omega^2 - \omega_0^2\right) + 2\Omega r\dot{\beta} & (51) \\ r\ddot{\beta} + 2\dot{r}\dot{\beta} &= -2\Omega\dot{r} \quad . & (52) \end{aligned}$$

This forces $\ddot{\beta}$ to vanish. In total the differential equation in polar representation is given by

$$\ddot{r} - r\dot{\beta}^2 = -r\omega_0^2 + 2\Omega r\dot{\beta} + r\Omega^2 \quad (53)$$

$$\ddot{\beta} = 0 \quad . \quad (54)$$

Taking the ansatz of circular orbits $r = \text{const}$, for simplicity $r = 1$ this reduces to

$$-\dot{\beta}^2 = -\omega_0^2 + 2\Omega\dot{\beta} + \Omega^2 \quad . \quad (55)$$

The ansatz $\beta = (\omega_0 - \Omega)\, t$ and $\beta = (-\omega_0 - \Omega)\, t$ solves the radial differential equation resulting in the same circular orbits – here with unity length radius – as given by (19) and (20) and we are done.

6 The Foucault Pendulum on Earth

A pendulum hanging at the North or South Pole of the Earth – we admit the inconvenience from the experimental point of view – can be modelled in the formalisms shown in the previous sections. In particular from the Earth view picture shown in section 5.1 it is evident, that a motion of the pendulum along the rotation axis is not affected by the Coriolis force since the cross product $\vec{\Omega} \times \vec{v}$ is vanishing where $\vec{v}$ is the pendulum bob velocity. Even more, a motion in a plane whose normal is oriented perpendicular to the rotation axis does not show Foucault rosettes. For sure such a motion is not free from inertial forces[2], but these do not affect the general trajectory shape and will not be taken into account in this letter. The Coriolis force gets smaller when we move from the poles towards the equator. The relevant – we can call it vertical – component of the angular velocity is only $\Omega \sin(\Theta)$ where Θ is the geographic latitude. To summarize, in zeroth order approximation here we have to replace Ω by an effective angular frequency $\Omega_{\text{eff}} = \Omega \sin(\Theta)$. The rest of the formalism remains unchanged. The influence of the non-inertial frame of reference to the oscillation itself is beyond this letter.

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[2]Think of a quickly spinning sphere and a slowly oscillating pendulum hanging at the equator.